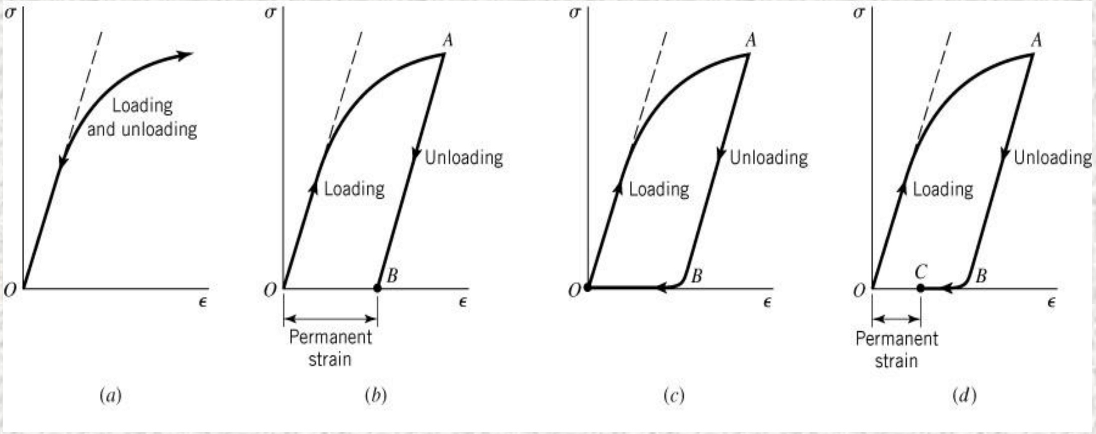
CHAPTER 4

Inelastic Material Behavior



EGM 5653
Advanced Mechanics of Materials

Objectives

Nonlinear material behavior

Yield criteria

Yielding in ductile materials

Sections

4.1 Limitations of Uniaxial Stress- Strain data

4.2 Nonlinear Material Response

4.3 Yield Criteria : General Concepts

4.4 Yielding of Ductile Materials

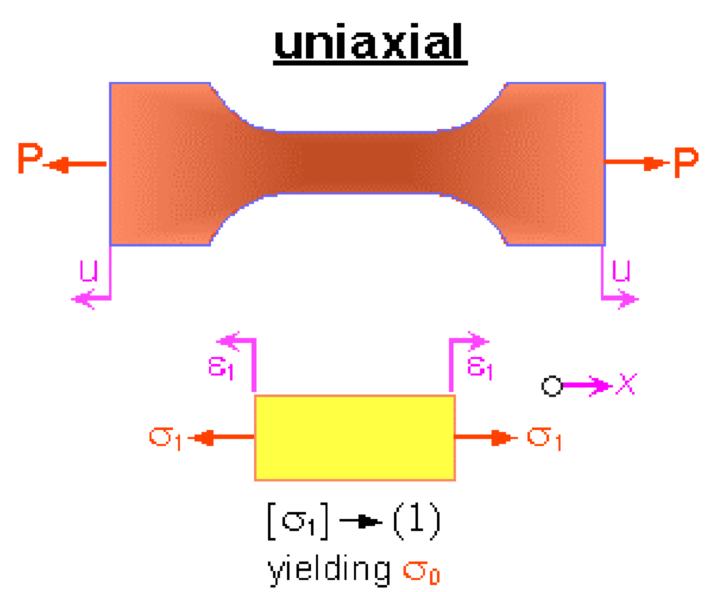
4.5 Alternative Yield Criteria

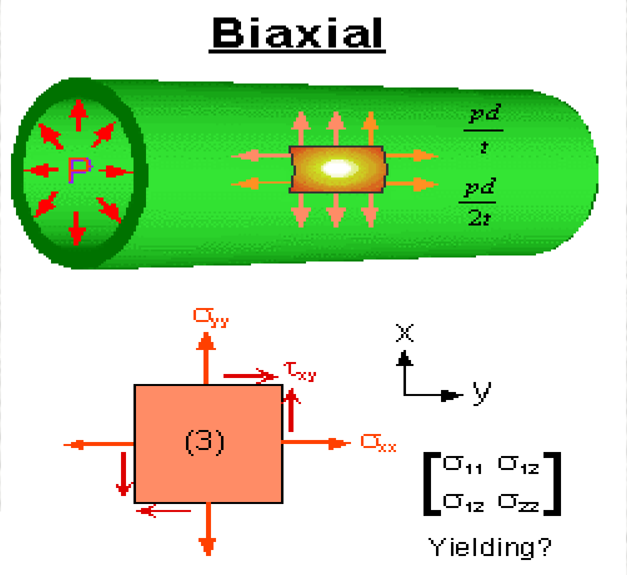
4.6 General Yielding

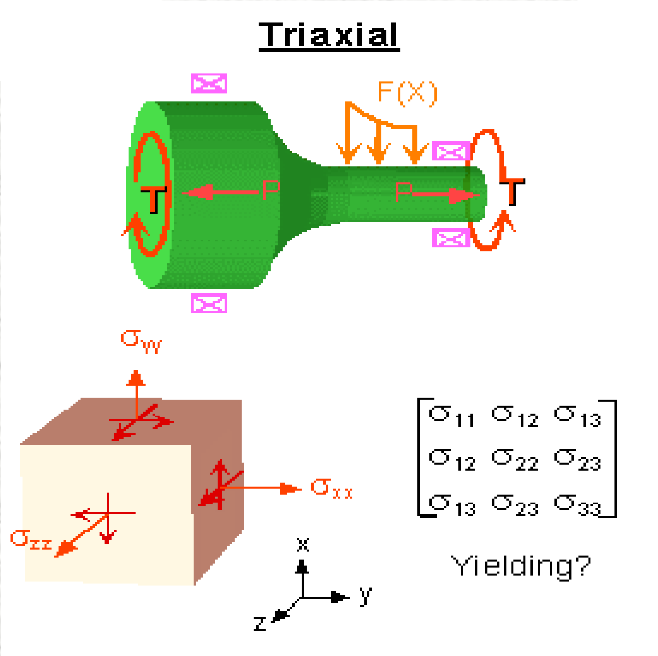
Introduction

- When a material is elastic, it returns to the same state (at macroscopic, microscopic and atomistic levels) upon removal of all external load
- Any material is not elastic can be assumed to be inelastic
  E.g.. Viscoelastic, Viscoplastic, and plastic
- To use the measured quantities like yield strength etc. we need some criteria
- The criterias are mathematical concepts motivated by strong experimental observations
  E.g. Ductile materials fail by shear stress on planes of maximum shear stress
- Brittle materials by direct tensile loading without much yielding
- Other factors affecting material behavior
  - Temperature
  - Rate of loading
  - Loading/ Unloading cycles

Types of Loading

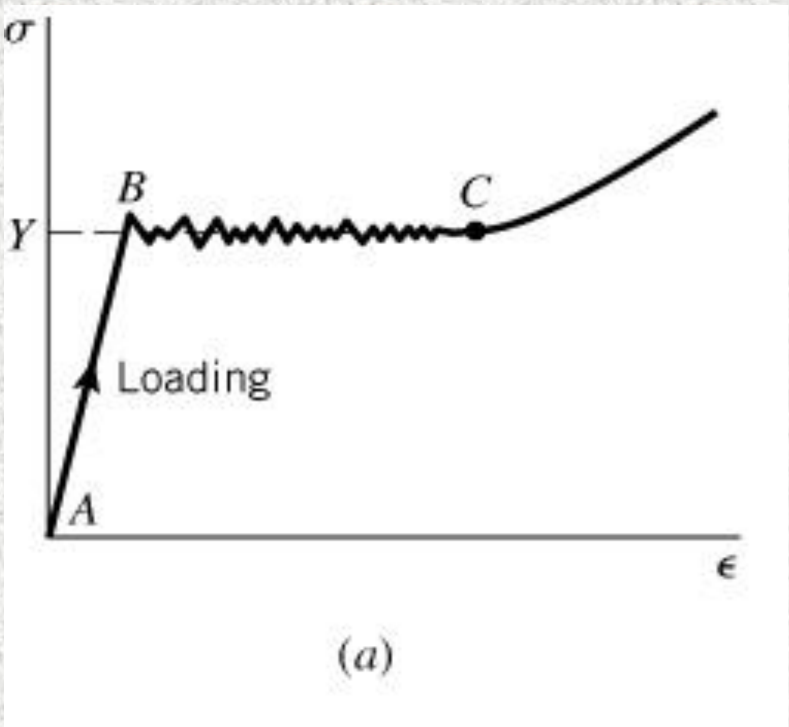




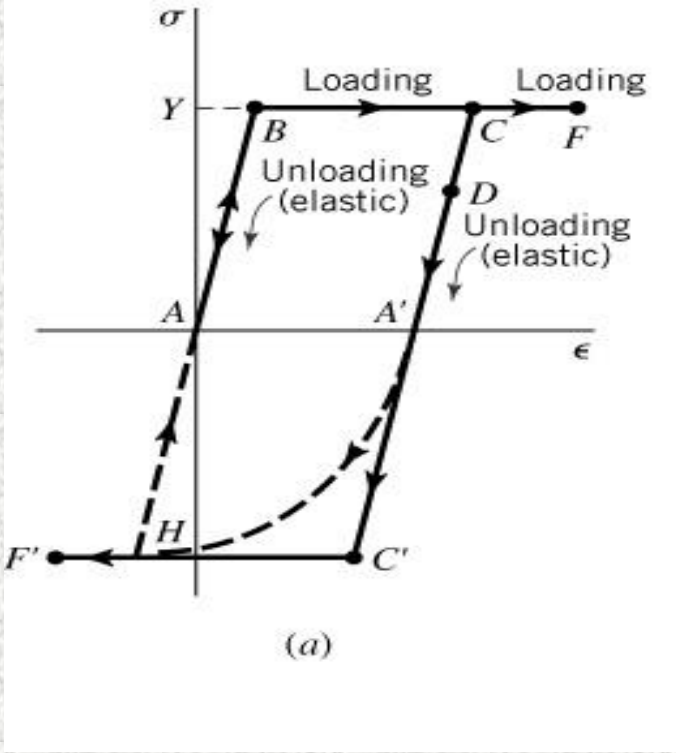


4.2.1 Models for Uniaxial stress-strain

All constitutive equations are models that are supposed to represent the physical behavior as described by experimental stress-strain response

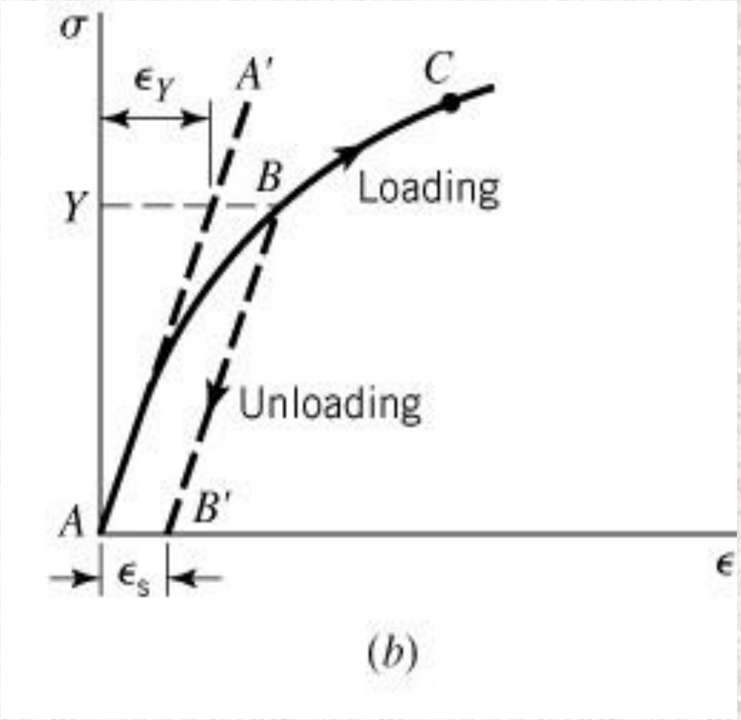


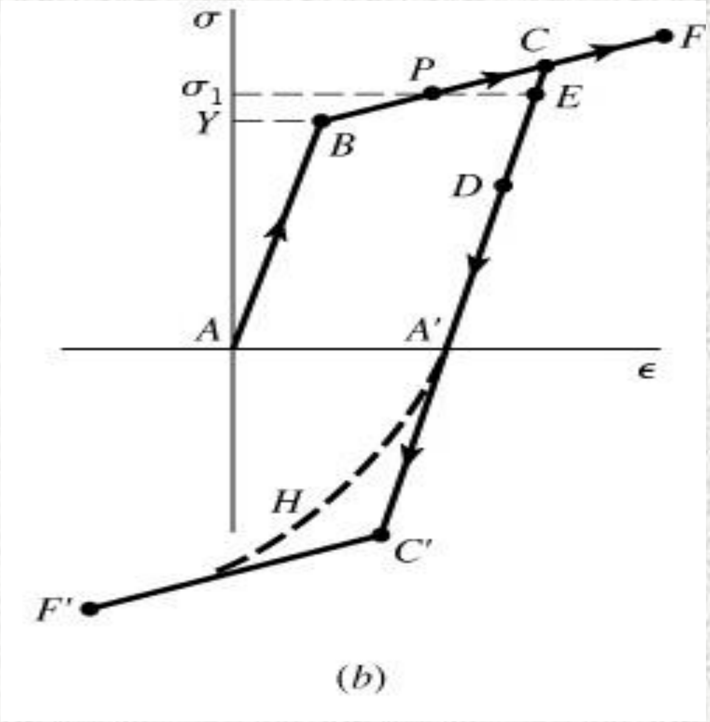
Experimental Stress strain curves



Idealized stress strain curves
Elastic- perfectly plastic response

4.2.1 Models for Uniaxial stress-strain contd.





Linear elastic response Elastic strain hardening response

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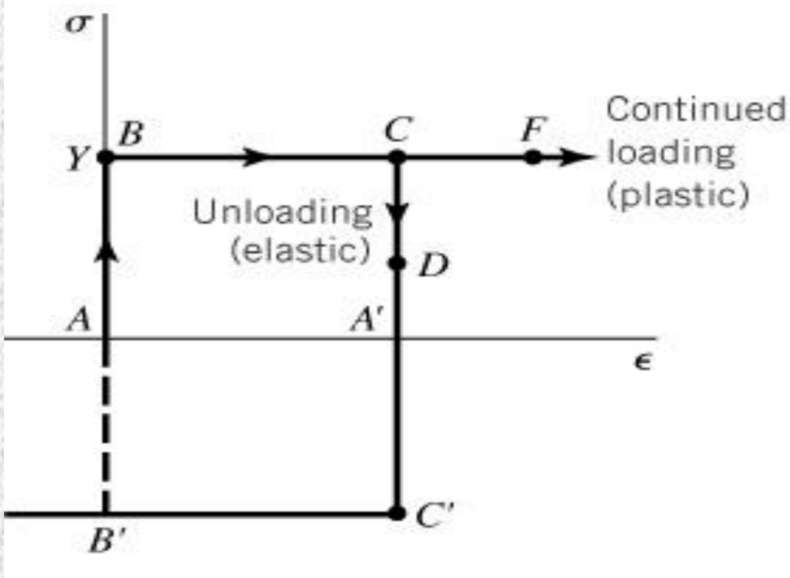
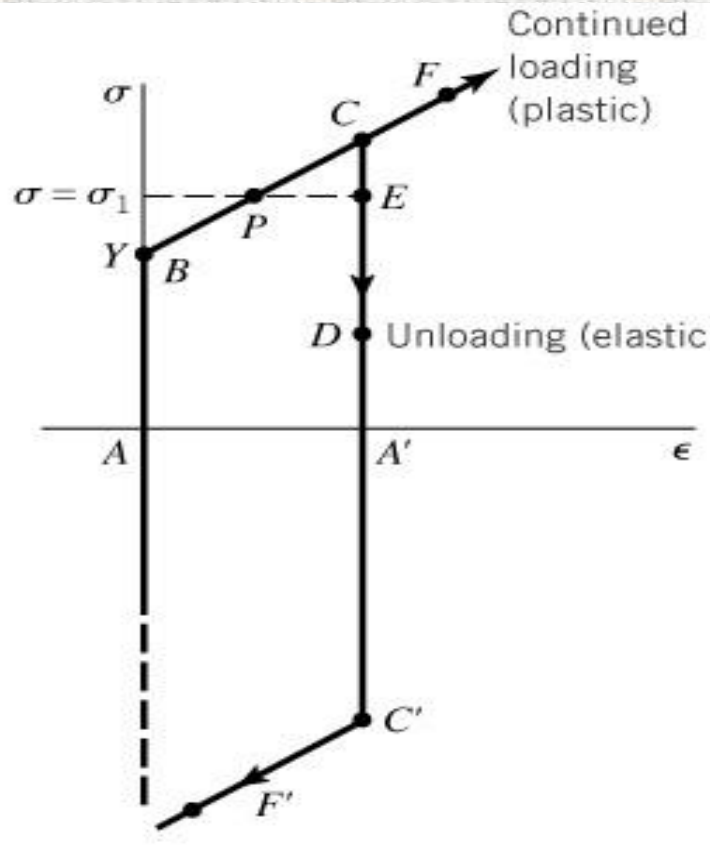
4.2.1 Models for Uniaxial stress-strain contd.

Rigid models



(a)

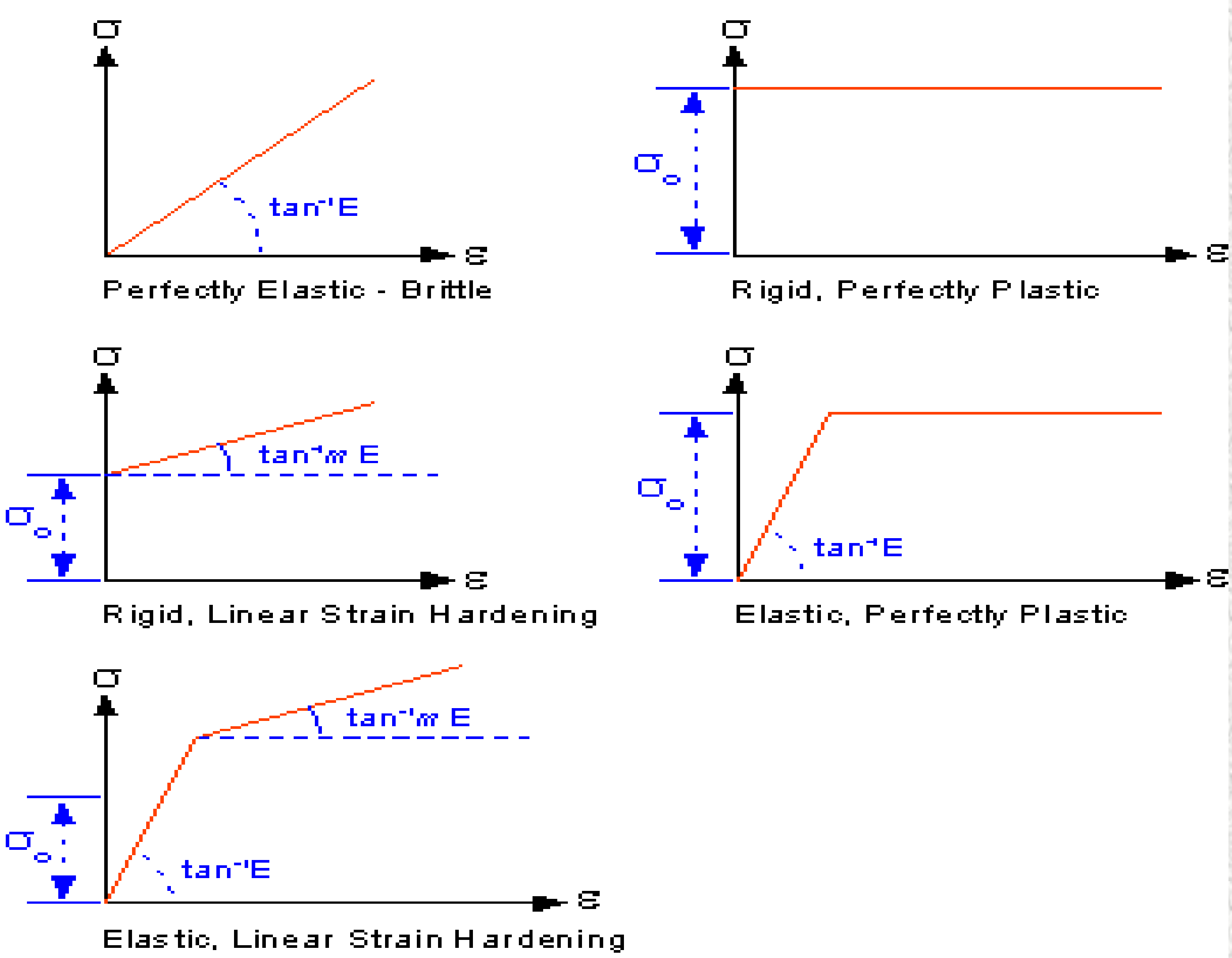
Rigid- perfectly plastic response



(b)

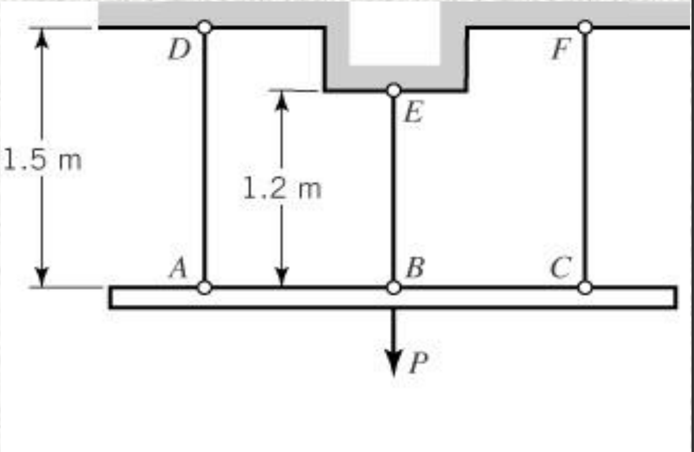
Rigid- strain hardening plastic response

Ideal Stress Strain Curves



4.2.1 Models for Uniaxial stress-strain contd.

4.4 The members AD and CF are made of elastic- perfectly plastic structural steel, and member BE is made of 7075 –T6 Aluminum alloy. The members each have a cross-sectional area of 100 mm^2.Determine the load P= P_Y that initiates yield of the structure and the fully plastic load P_P for which all the members yield.



Soln:

4.4 Let subscript S denote steel and subcript A denote aluminum. Then from Appendix A, $E_S = 200$ GPa, $Y_S = 250$ MPa, $E_A = 72$ MPa, and $Y_A = 500$ MPa. For a given vertical displacement u of beam ABC, the strains in the steel and aluminum bars are, respectively, $\epsilon_S = u/L_S$ and $\epsilon_A = u/L_A$. Thus, the stresses in the bars are

$$\sigma_S = E_S \epsilon_S = E_S u/L_S \quad \text{and} \quad \sigma_A = E_A \epsilon_A = E_A u/L_A .$$

Contd..

4.2.1 Models for Uniaxial stress-strain contd.

4.4 continued: For yield of the steel bars, $u = Y_S L_S / E_S$ or $u = 250(1.5)/200 = 1.875$ mm. For yield of the aluminum bar $u = Y_A L_A / E_A = 500(1.2)/72 = 8.333$ mm. Therefore, the steel bars yield first. With $u = 1.875$ mm, the stress in the steel bars is $Y_S = 250$ MPa, and the stress in the aluminum bar is $\sigma_A = E_A u / L_A = \frac{72(1.875)}{1.2} = 112.5$ MPa

(a) At yield of the steel bars, summation of forces in the vertical direction yields the result

$$P = P_Y = 2 Y_S A + \sigma_A A = [2(250) + 112.5](100) = 61.25 \text{ kN}$$

(b) Similarly at yield of the aluminum bar,

$$P = P_P = 2 Y_S A + Y_A A = [2(250) + 500](100) = 100 \text{ kN}$$

4.3 The Yield Criteria : General concepts

General Theory of Plasticity defines

Yield criteria : predicts material yield under multi-axial state of stress

Flow rule : relation between plastic strain increment and stress increment

Hardening rule: Evolution of yield surface with strain

Yield Criterion is a mathematical postulate and is defined by a yield function $f = f(\{\sigma_{ij}\}, Y)$

where Y is the yield strength in uniaxial load, and is correlated with the history of stress state.

Some Yield criteria developed over the years are:

Maximum Principal Stress Criterion:- used for brittle materials

Maximum Principal Strain Criterion:- sometimes used for brittle materials

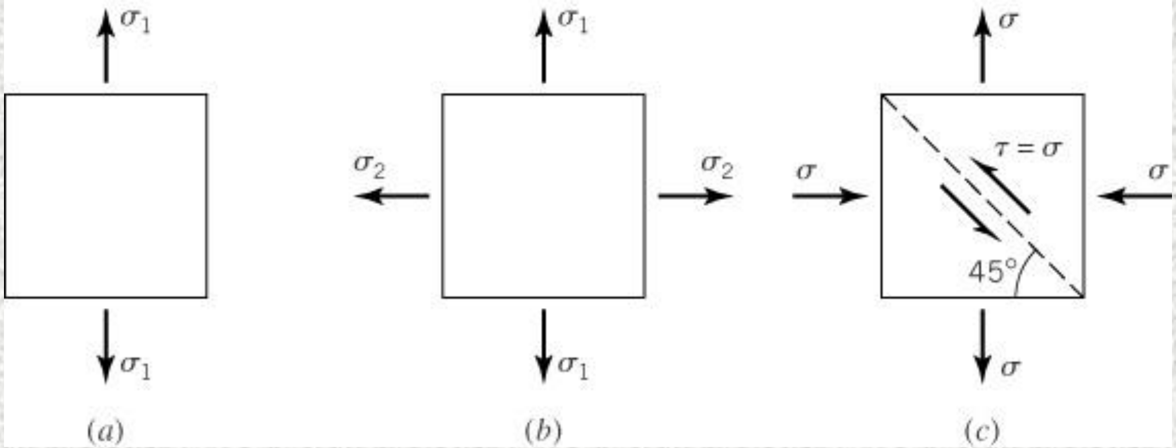
Strain energy density criterion:- ellipse in the principal stress plane

Maximum shear stress criterion (a.k.a Tresca):- popularly used for ductile materials

Von Mises or Distortional energy criterion:- most popular for ductile materials

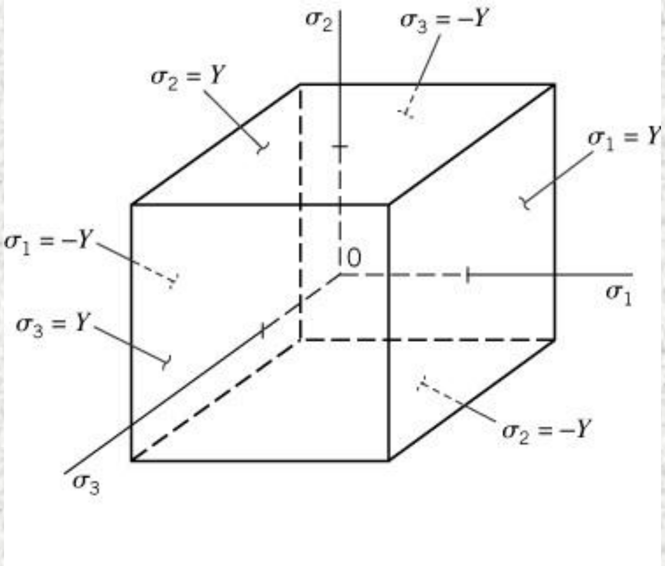
4.3.1 Maximum Principal Stress Criterion

Originally proposed by Rankine



$$Y_C = \frac{2c\cos\phi}{1-\sin\phi}$$

$$f = \max\left(|\sigma_1|,|\sigma_2|,|\sigma_3|\right) - Y$$



Yield surface is:

$$\sigma_1 = \pm Y$$

$$\sigma_2 = \pm Y$$

$$\sigma_3 = \pm Y$$

4.3.2 Maximum Principal Strain

This was originally proposed by St. Venant

$$f_1 = |\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3| - Y = 0$$ or $$\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3 = \pm Y$$

$$f_2 = |\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3| - Y = 0$$ or $$\sigma_2 - \upsilon\sigma_1 - \upsilon\sigma_3 = \pm Y$$

$$f_3 = |\sigma_3 - \upsilon\sigma_1 - \upsilon\sigma_2| - Y = 0$$ or $$\sigma_3 - \upsilon\sigma_1 - \upsilon\sigma_2 = \pm Y$$

Hence the effective stress may be defined as

$$\sigma_e = \max_{i \neq j \neq k} |\sigma_i - \upsilon\sigma_j - \upsilon\sigma_k|$$

The yield function may be defined as

$$f = \sigma_e - Y$$

4.3.2 Strain Energy Density Criterion

This was originally proposed by Beltrami

Strain energy density is found as

$$U_0 = \frac{1}{2E}\left[\sigma_1^{\ 2} + \sigma_1^{\ 2} + \sigma_1^{\ 2} - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right] > 0$$

Strain energy density at yield in uniaxial tension test

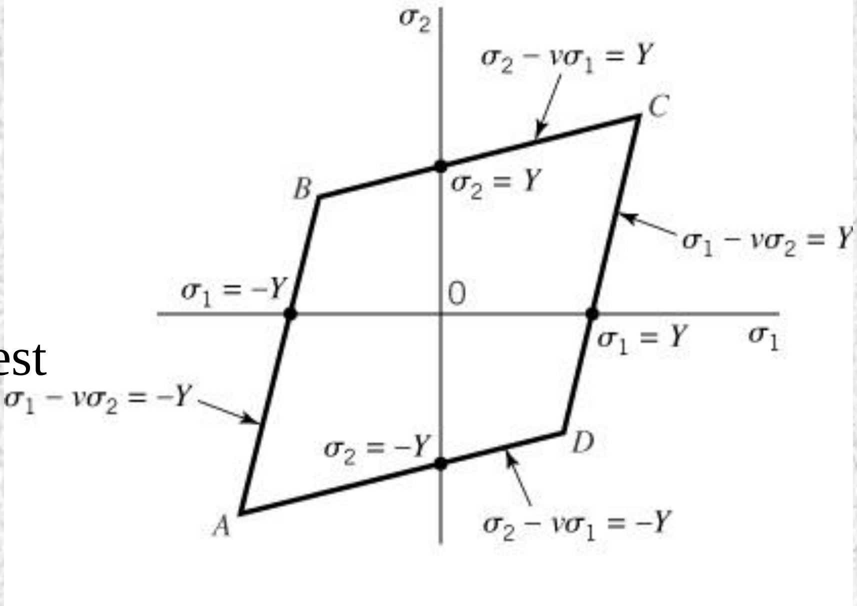
$$U_{0Y} = \frac{Y^2}{2E}$$

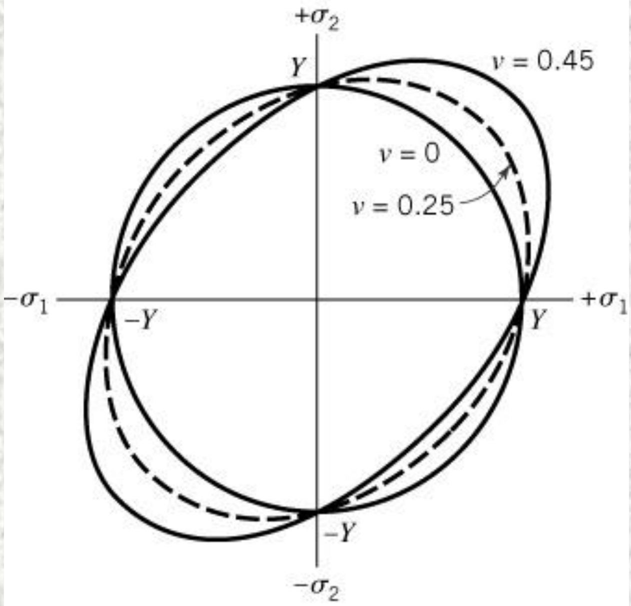
Yield surface is given by

$$\sigma_1^{\ 2} + \sigma_1^{\ 2} + \sigma_1^{\ 2} - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) - Y^2 = 0$$

$$f = \sigma_e^2 - Y^2$$

$$\sigma_e = \sqrt{\sigma_1^{\ 2} + \sigma_1^{\ 2} + \sigma_1^{\ 2} - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)}$$





4.4.1 Maximum Shear stress (Tresca) Criterion

This was originally proposed by **Tresca**

Yield function is defined as

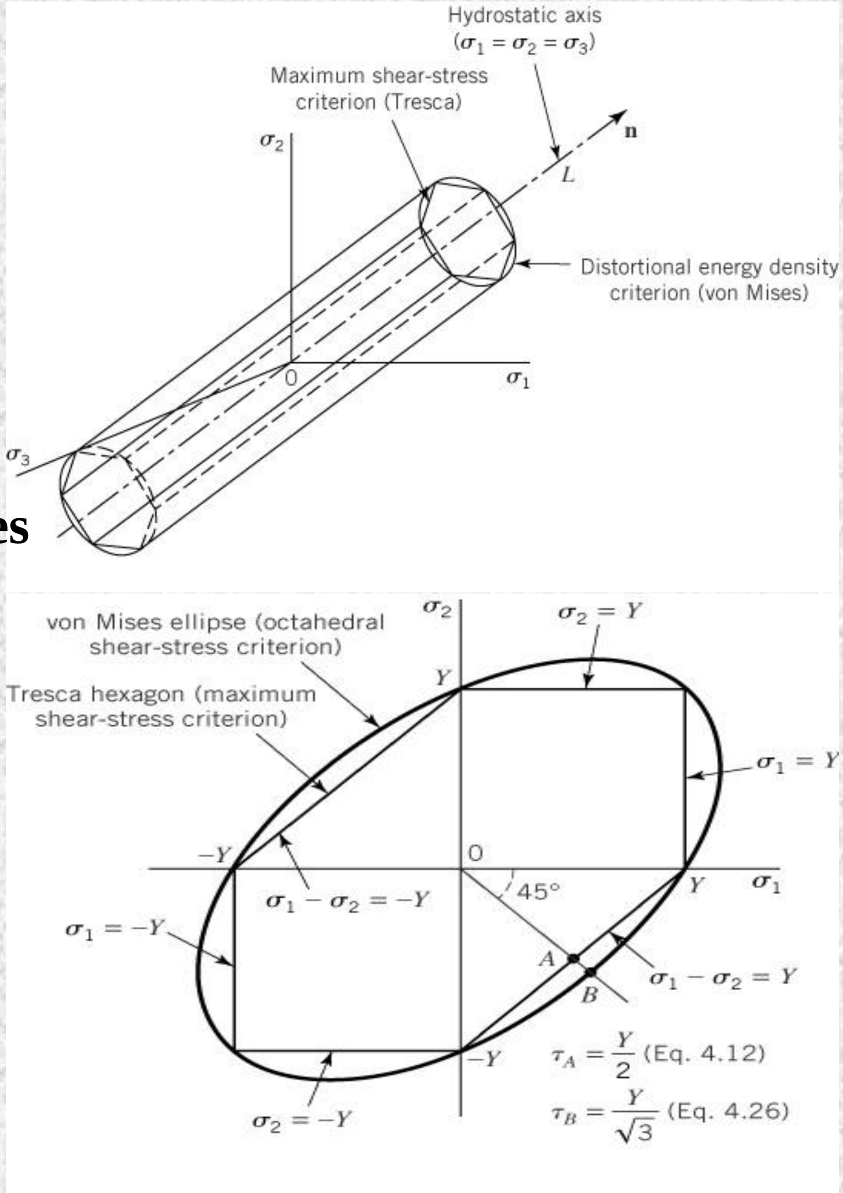
$$f = \sigma_e - \frac{Y}{2}$$

where the **effective stress** is

$$\sigma_e = \tau_{\max}$$

Magnitude of the **extreme values of the stresses** are

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2}$$
$$\tau_2 = \frac{|\sigma_3 - \sigma_1|}{2}$$
$$\tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$



- Conditions in which **yielding** can occur in a **multi-axial stress state**

$$\sigma_2 - \sigma_3 = \pm Y$$
$$\sigma_3 - \sigma_1 = \pm Y$$
$$\sigma_1 - \sigma_2 = \pm Y$$

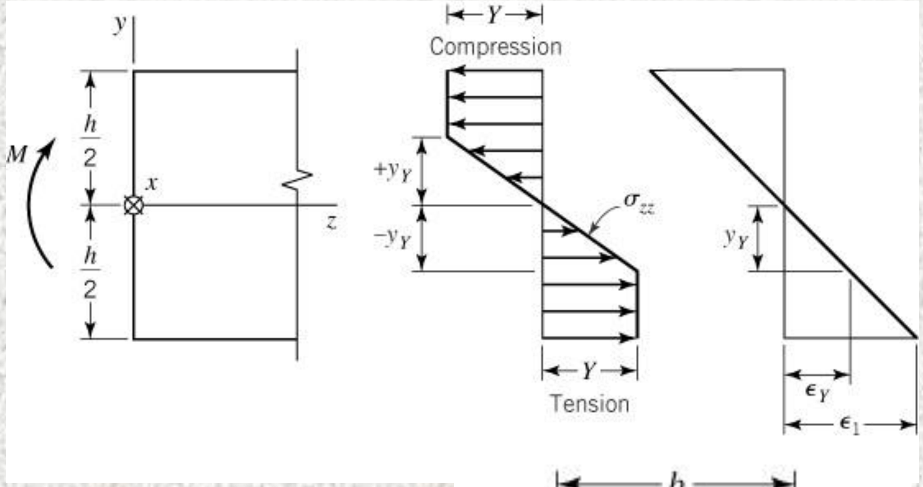
4.4.2 Distortional Energy Density (von Mises) Criterion

Originally proposed by **von Mises** & is the most popular for **ductile materials**

Total strain energy density = SED due to volumetric change +SED due to distortion

$$U_0 = \frac{(\sigma_1 - \sigma_2 - \sigma_3)^2}{18} + \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{12G}$$

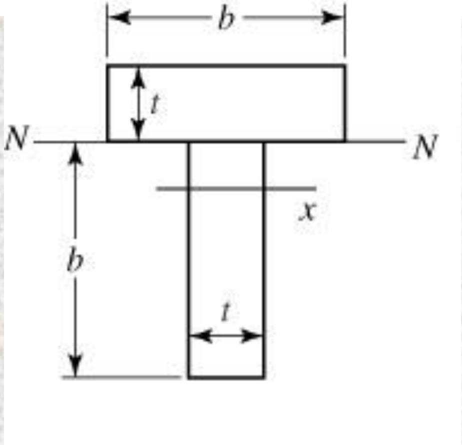
$$U_D = \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{12G}$$



The yield surface is given by

$$|J_2| = \frac{1}{3} Y^2$$

$$f = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] - \frac{1}{3} Y^2$$



(a)

4.4.2 Distortional Energy Density (von Mises) Criterion contd.

Alternate form of the yield function

$$f = \sigma_e^{\,2} - Y^2$$

where the **effective stress** is

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} = \sqrt{3|J_2|}$$

$$\sigma_e = \sqrt{\frac{1}{2}\left[\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \left(\sigma_{yy} - \sigma_{zz}\right)^2 + \left(\sigma_{zz} - \sigma_{xx}\right)^2\right] + 3\left(\sigma_{xy}^{\,2} + \sigma_{yz}^{\,2} + \sigma_{xz}^{\,2}\right)}$$

J_2 and the octahedral shear stress are related by

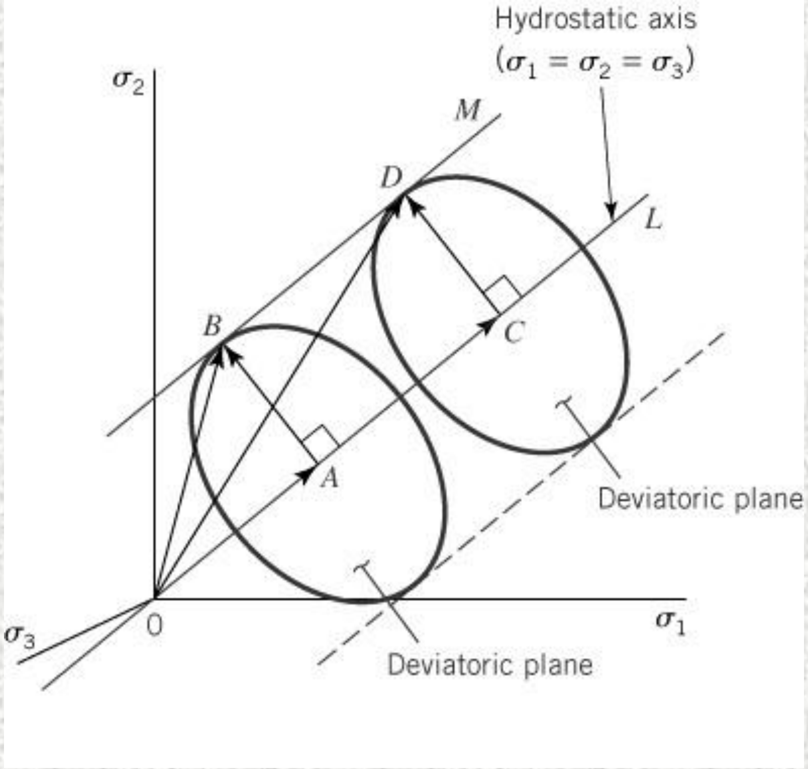
$$J_2 = -\frac{3}{2}\tau_{oct}^2$$

Hence the von Mises yield criterion can be written as

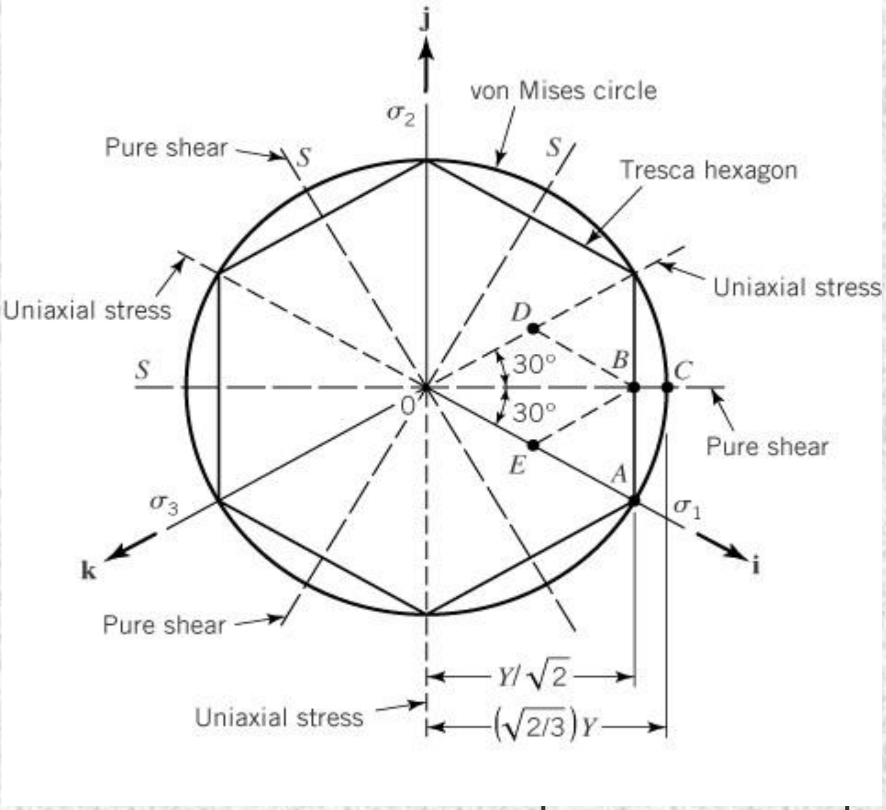
$$f = \tau_{oct} - \frac{\sqrt{2}}{3}Y$$

4.4.3 Effect of Hydrostatic stress and the π- plane

Hydrostatic stress has no influence on yielding



Definition of a π- plane



4.5 Alternate Yield Criteria

Generally used for non ductile materials like rock, soil, concrete and other anisotropic materials

4.5.1 Mohr-Coloumb Yield Criterion

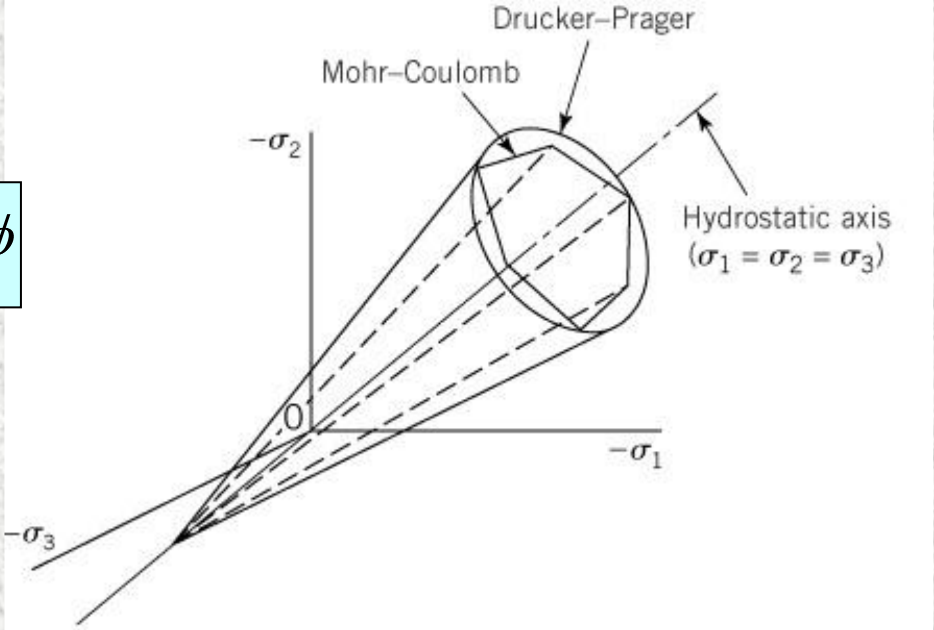
- ❖ Very useful for rock and concretes
- ❖ Yielding depends on the hydrostatic stress

$$f = \sigma_1 - \sigma_3 + \left(\sigma_1 + \sigma_3\right)\sin\phi - 2c\cos\phi$$

$$f = \max_{i \neq j}\left[\sigma_i - \sigma_j + \left(\sigma_i + \sigma_j\right)\sin\phi\right] - 2c\cos\phi$$

$$Y_T = \frac{2c\cos\phi}{1+\sin\phi}$$

$$Y_C = \frac{2c\cos\phi}{1-\sin\phi}$$



(a)

4.5.2 Drucker-Prager Yield Criterion

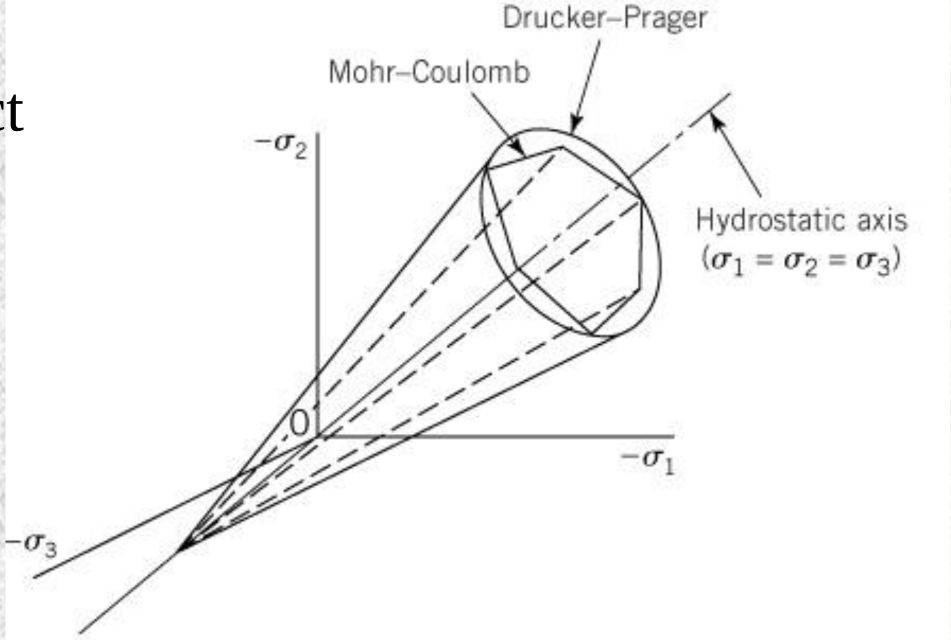
This is the generalization of von Mises criteria with the hydrostatic stress effect included

Yield function can be written as

$$f = \alpha I_1 + \sqrt{|J_2|} - K$$

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3-\sin\phi)},\ K = \frac{6c\cos\phi}{\sqrt{3}(3-\sin\phi)}$$

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3+\sin\phi)},K = \frac{6c\cos\phi}{\sqrt{3}(3+\sin\phi)}$$



(a)

4.5.3 Hill's Yield Criterion for Orthotropic Materials

This is the criterion is used for non-linear materials

The yield function is given by

$$f = F\left(\sigma_{22}-\sigma_{33}\right)^2 + G\left(\sigma_{33}-\sigma_{11}\right)^2 + H(\sigma_{11}-\sigma_{22})^2$$
$$+ L(\sigma_{23}^2+\sigma_{32}^2) + M(\sigma_{13}^2+\sigma_{31}^2) + N(\sigma_{12}^2+\sigma_{21}^2) - 1$$

$$2F = \frac{1}{Z^2}+\frac{1}{Y^2}-\frac{1}{X^2}$$
$$2G = \frac{1}{Z^2}+\frac{1}{X^2}-\frac{1}{Y^2}$$
$$2H = \frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}$$
$$2L = \frac{1}{S_{23}^2},\ 2M = \frac{1}{S_{13}^2},\ 2N = \frac{1}{S_{12}^2}$$

For an isotropic material

$$6F = 6G = 6H = L = M = N$$

General Yielding

The failure of a material is when the structure cannot support the intended function

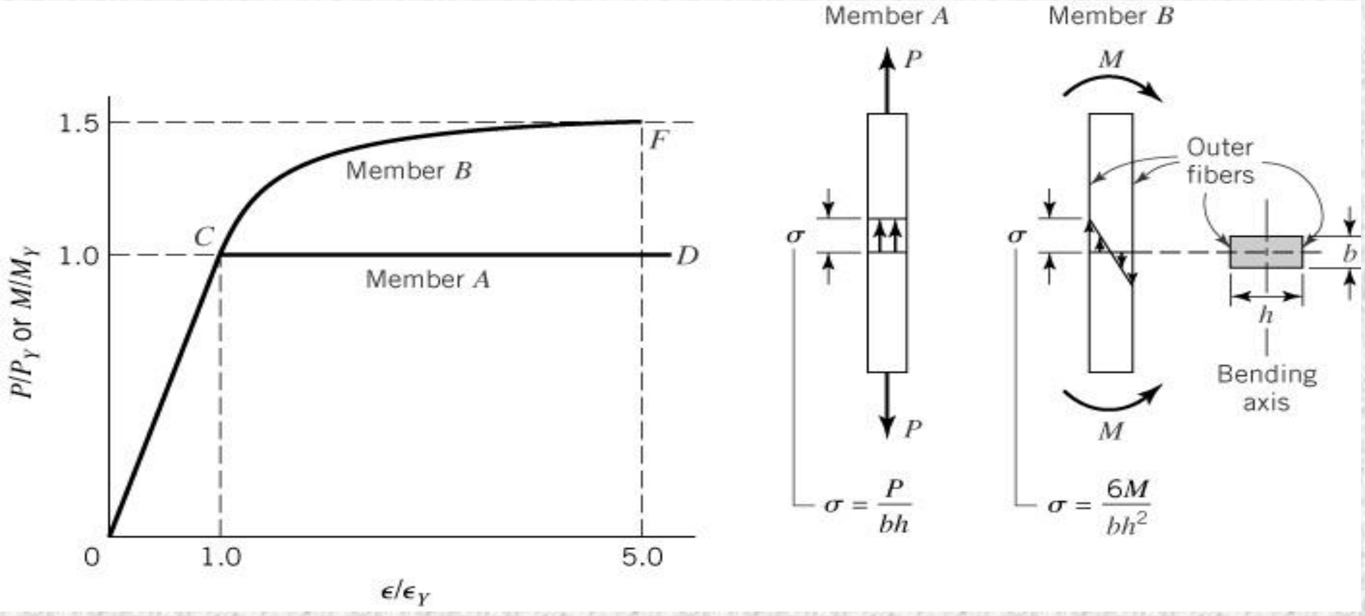
For some special cases, the loading will continue to increase even beyond the initial load

At this point, part of the member will still be in elastic range. When the entire member reaches the inelastic range, then the general yielding occurs

$$P_Y = Ybh,\ M_Y = Y\frac{bh^2}{6}$$

$$P_P = Ybh = P_Y$$

$$M_P = Y\frac{bh^2}{4} = 1.5M_Y$$



4.6.1 Elastic Plastic Bending

Consider a beam made up of elastic-perfectly plastic material subjected to bending. We want to find the maximum bending moment the beam can sustain

$$\varepsilon_{zz} = \varepsilon_1 = k\varepsilon_Y \qquad (a)$$

where,

$$\varepsilon_Y = \frac{Y}{E} \qquad (b)$$

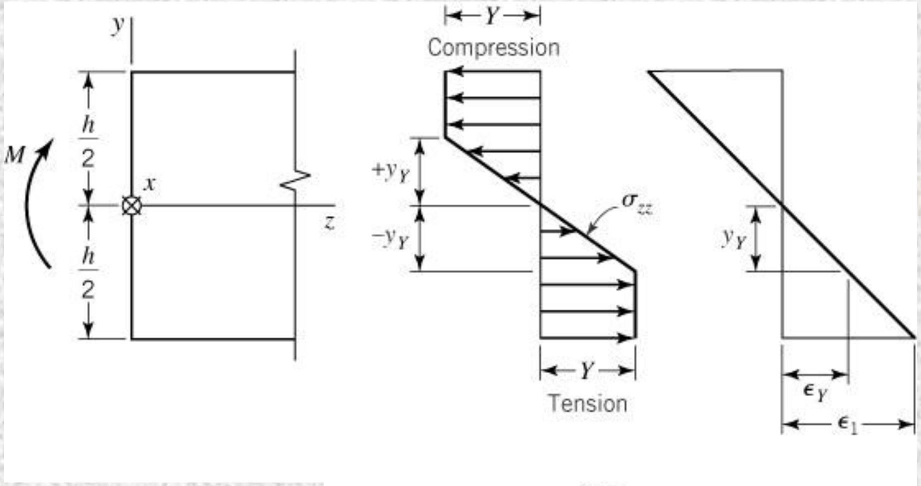
$$y_Y = \frac{h}{2k} \qquad (c)$$

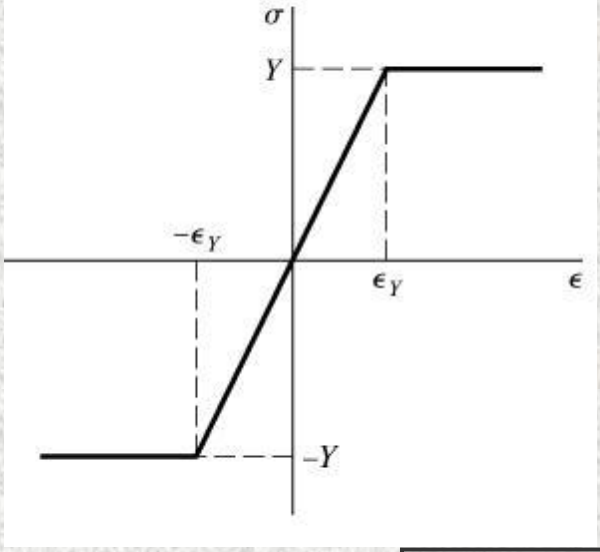
$$\sum F_Z = \int \sigma_{zz} dA = 0 \qquad (d)$$

$$\sum M_x = M - 2\int_0^{y_Y} \sigma_{ZZ}\, y dA - 2\int_{y_Y}^{h/2} Y_y dA = 0$$

or

$$M = M_{EP} = 2\int_0^{y_Y} \sigma_{zz}\, y dA + 2Y\int_{y_Y}^{h/2} y dA \qquad (e)$$





4.6.1 Elastic Plastic Bending contd.

$$M_{EP} = \frac{Ybh^2}{6}\left(\frac{3}{2} - \frac{1}{2k^2}\right) = M_Y\left(\frac{3}{2} - \frac{1}{2k^2}\right) \qquad (4.43)$$

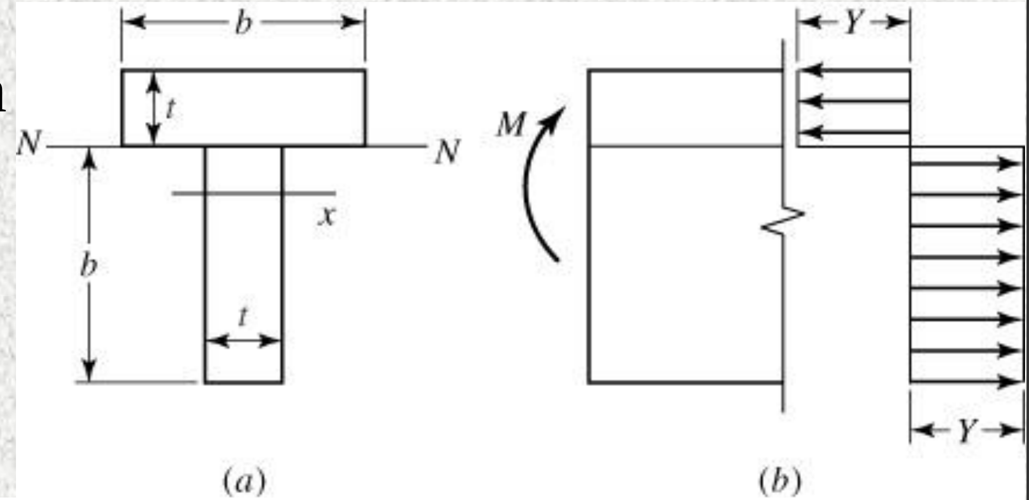
$$where, M_Y = Ybh^2/6$$

as k becomes large

$$M_{EP} \rightarrow \frac{3}{2} M_Y = M_P$$

4.6.2 Fully Plastic Bending

Definition: Bending required to cause yielding either in tension or compression over the entire cross section

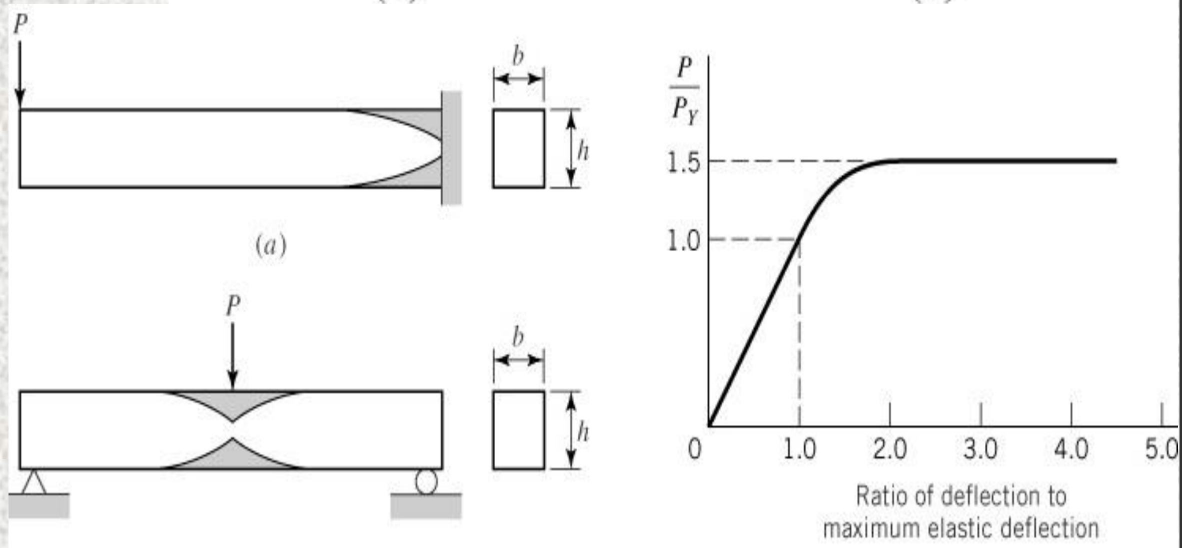


Equilibrium condition

$$\sum F_z = \int \sigma_{zz} dA = 0$$

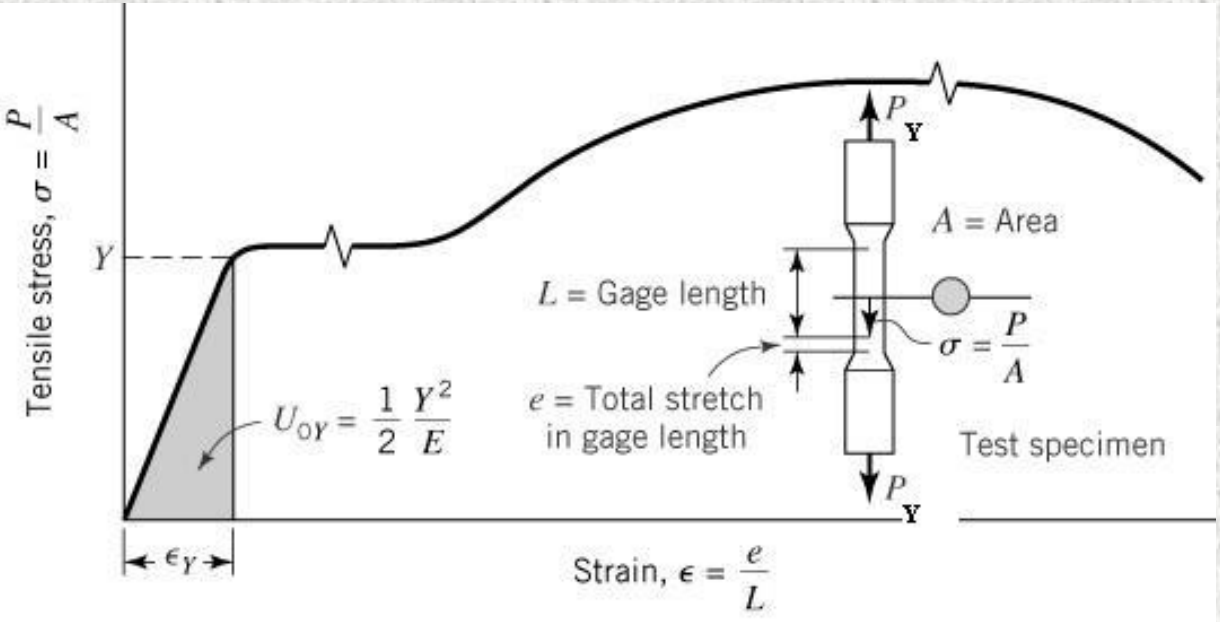
Fully plastic moment is

$$M_P = Ybt\left(\frac{t+b}{2}\right)$$



Comparison of failure yield criteria

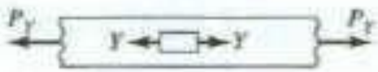
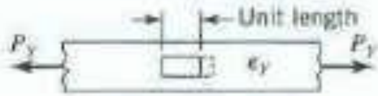
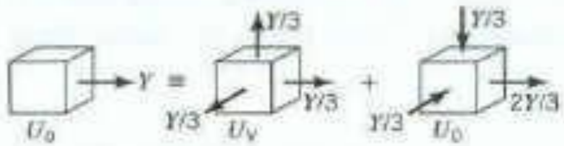
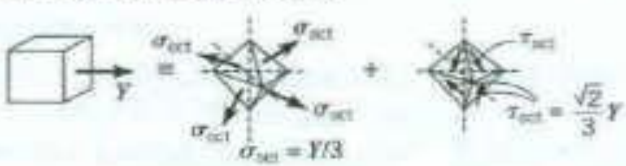
For a tensile specimen of ductile steel the following six quantities attain their critical values at the same load P_Y



1. Maximum principal stress $(\sigma_{max} = P_Y / A)$ reaches the yield strength Y
2. Maximum principal strain $(\varepsilon_{max} = \sigma_{max} / E)$ reaches the value $\varepsilon_Y = Y / E$
3. Strain energy Uo absorbed by the material per unit volume reaches the value $U_{0Y} = Y^2 / 2E$
4. The maximum shear stress $(\tau_{max} = P_Y / 2A)$ reaches the tresca shear strength $(\tau_Y = Y / 2)$
5. The distortional energy density U_D reaches $U_{DY} = Y^2 / 6G$
6. The octahedral shear stress $\tau_{oct} = \sqrt{2}Y / 3 = 0.471Y$

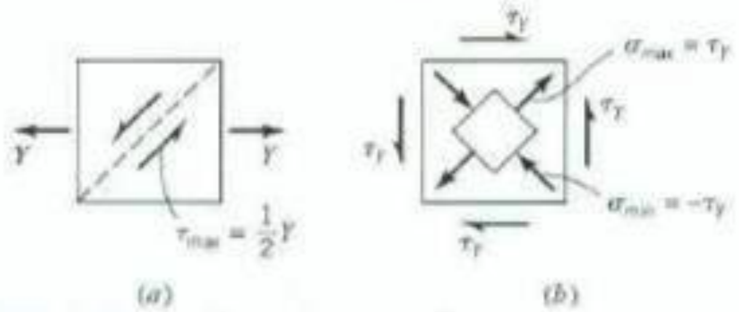
Failure criteria for general yielding

TABLE 4.1 Failure Criteria for General Yielding

Quantity	Critical value in terms of tension test
1. Maximum principal stress 	$Y = P_Y/A$
2. Maximum principal strain 	$\epsilon_Y = Y/E$
3. Strain-energy density 	$U_{0Y} = Y^2/2E$
4. Maximum shear stress 	$\tau_Y = P_Y/2A = Y/2$
5. Distortional energy density 	$U_{DY} = \frac{Y^2}{6G}, \quad G = \frac{E}{2(1+\nu)}$
6. Octahedral shear stress 	$\tau_{oct} = (\sqrt{2}/3)Y = 0.471Y$

Interpretation of failure criteria for general yielding

TABLE 4.2 Comparison of Maximum Utilizable Values of a Material Quantity According to Various Yield Criteria for States of Stress in the Tension (a) and Torsion (b) Tests



(a) (b)

(1) Yield criterion	(2) Predicted maximum utilizable value as obtained from a tension test (a)	(3) Predicted maximum utilizable value as obtained from a torsion test (b)	(4) Relation between values of Y and τ_Y if the criterion is correct for both stress states (col. 2 = col. 3)
Maximum principal stress	$\sigma_{max} = Y$	$\sigma_{max} = \tau_Y$	$\tau_Y = Y$
Maximum principal strain, $\nu = 1/4$	$\epsilon_{max} = \frac{Y}{E}$	$\epsilon_{max} = \frac{5}{4}\frac{\tau_Y}{E}$	$\tau_Y = \frac{4}{5}Y$
Maximum shear stress	$\tau_{max} = \frac{1}{2}Y$	$\tau_{max} = \tau_Y$	$\tau_Y = \frac{1}{2}Y$
Maximum octahedral shear stress	$\tau_{octY} = \frac{\sqrt{2}}{3}Y$	$\tau_{octY} = \sqrt{\frac{2}{3}}\tau_Y$	$\tau_Y = \frac{1}{\sqrt{3}}Y$
Maximum distortional energy density	$U_{DY} = \frac{Y^2}{6G}$	$U_{DY} = \frac{\tau_Y^2}{2G}$	$\tau_Y = \frac{1}{\sqrt{3}}Y$

Combined Bending and Loading

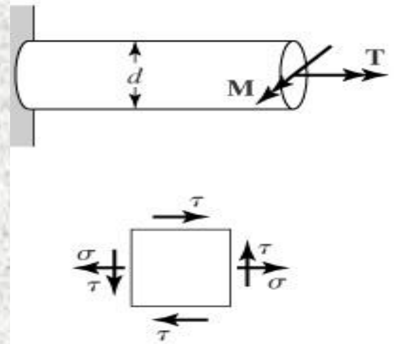
According to Maximum shear stress criteria, yielding starts when

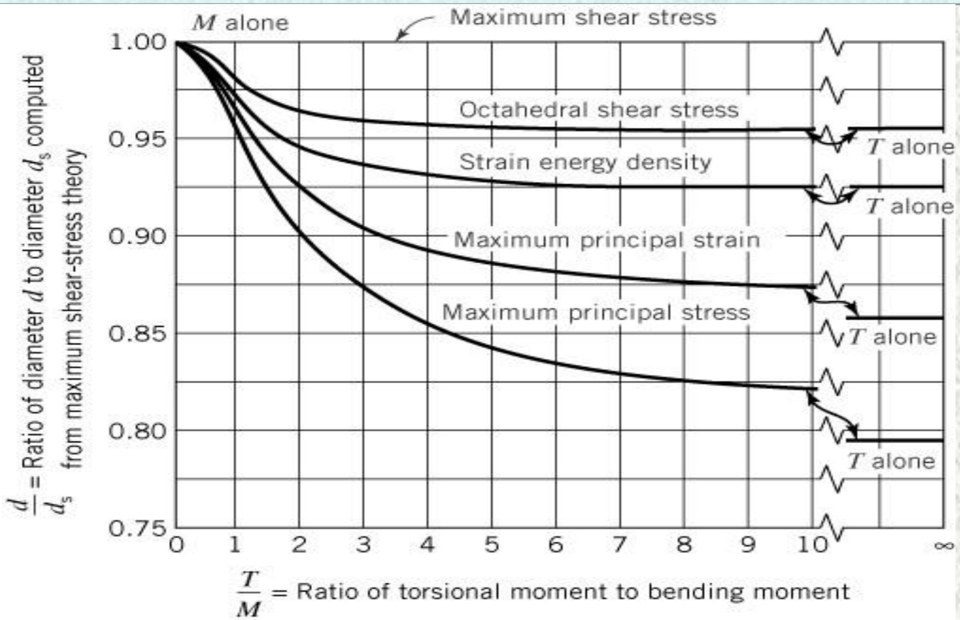
$$\sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{Y}{2} \text{ or } \left(\frac{\sigma}{2}\right)^2 + 4\left(\frac{\tau}{Y}\right)^2 = 1$$

According to the octahedral shear-stress criterion, yielding starts when

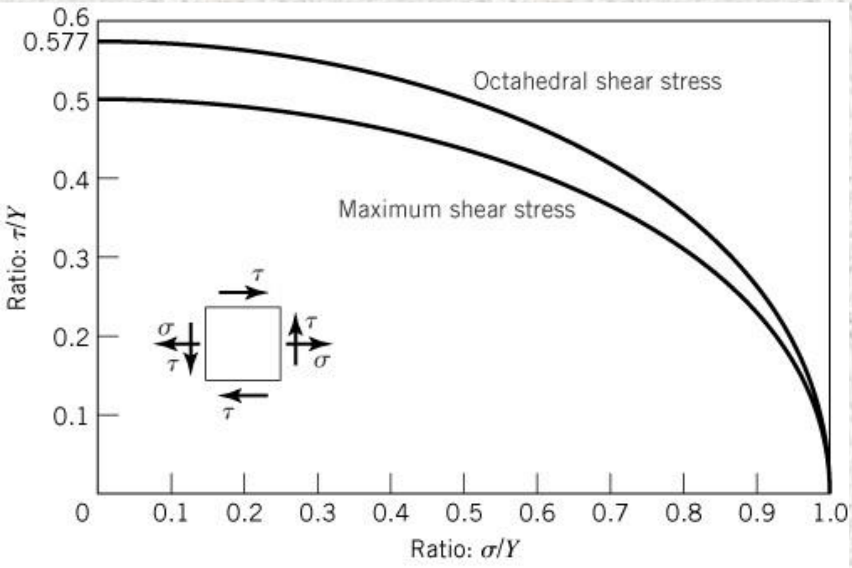
$$\sqrt{\frac{2\sigma^2 + 6\tau^2}{3}} = \frac{\sqrt{2}Y}{3} \text{ or } \left(\frac{\sigma}{Y}\right)^2 + 3\left(\frac{\tau}{Y}\right)^2 = 1$$

Interpretation of failure criteria for general yielding





Comparison of von Mises and Tresca criteria



Problem 4.24

4.24 A rectangular beam of width b and depth h is subjected to pure bending with a moment $M=1.25M_y$. Subsequently, the moment is released. Assume the plane sections normal to the neutral axis of the beam remain plane during deformation.

a. Determine the radius of curvature of the beam under the applied bending moment M=1.25My

b. Determine the distribution of residual bending stress after the applied bending moment is released

Solution:

4.24 (a) The moment is (see Prob. 4.23, with $\beta = 1.25$)

$$M = 1.25 M_y \qquad (a)$$

The maximum elastic moment is

$$M_y = \frac{Y I_x}{h/2} = \frac{1}{6} b h^2 Y \qquad (b)$$

By Eqs. (a) and (b),

$$M = \frac{5}{24} b h^2 Y \qquad (c)$$

By Fig. a, $M = \Sigma M_0 = 2\left(\frac{h}{2} - e\right) b Y \left(\frac{h/2 + e}{2}\right) = \left(\frac{h^2}{4} - \frac{1}{3} e^2\right) b Y \qquad (d)$

Problem 4.24 contd.

Equating Eqs. (c) and (d) we find

$$e = \frac{1}{\sqrt{2}}\frac{h}{2} \quad (e)$$

Since plane sections remain plane, the strain ϵ is a linear function of y (Fig. b). at $y = e$, $\epsilon = Y/E$, where E is the modulus of elasticity of the beam. The curvature of the beam is

$$\frac{1}{\rho} = \frac{\epsilon}{e} = \frac{Y}{Ee},$$

or the radius of curvature is

$$\rho = \frac{Ee}{Y}$$

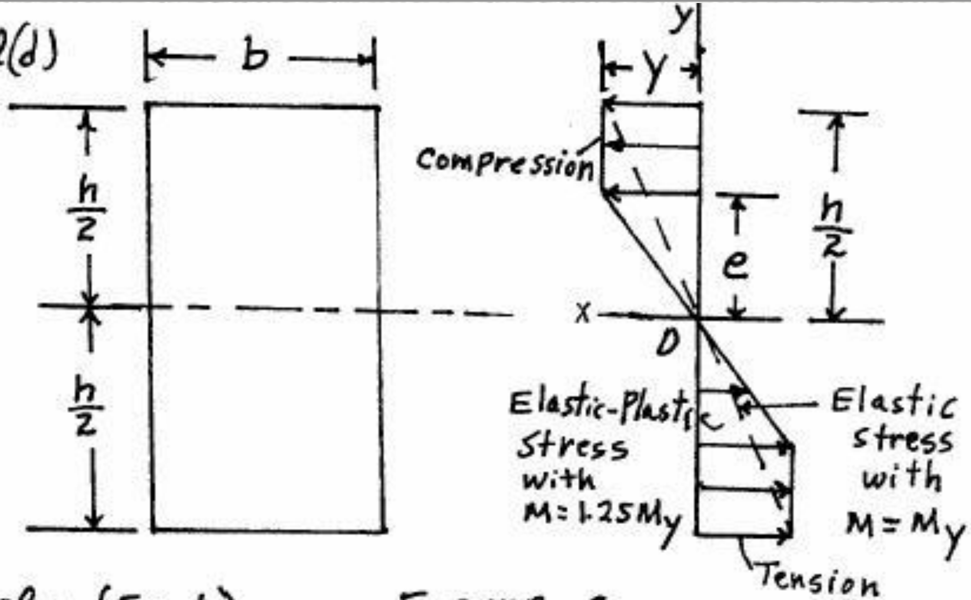


Figure a

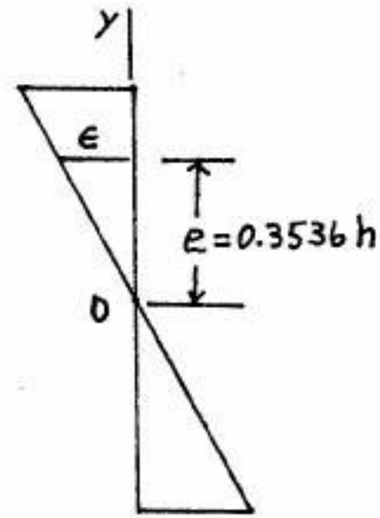


Figure b

(Cont.)

Problem 4.24 contd.

4.24 Cont. (b) With $M = 1.25M_Y$, the elastic stress upon release is

$$\sigma_E = -\frac{My}{I_x} = -\frac{\frac{5}{4}\left(\frac{1}{6}bh^2Y\right)y}{\frac{1}{12}bh^3} = -\frac{5Y}{2}\left(\frac{y}{h}\right)$$

Subtracting σ_E from the elastic-plastic stress distribution, we obtain the residual stress distribution shown in Fig. c.

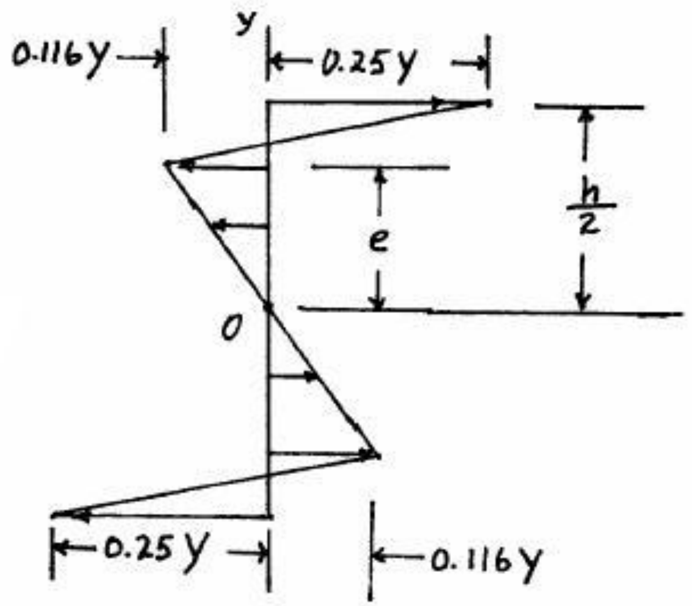
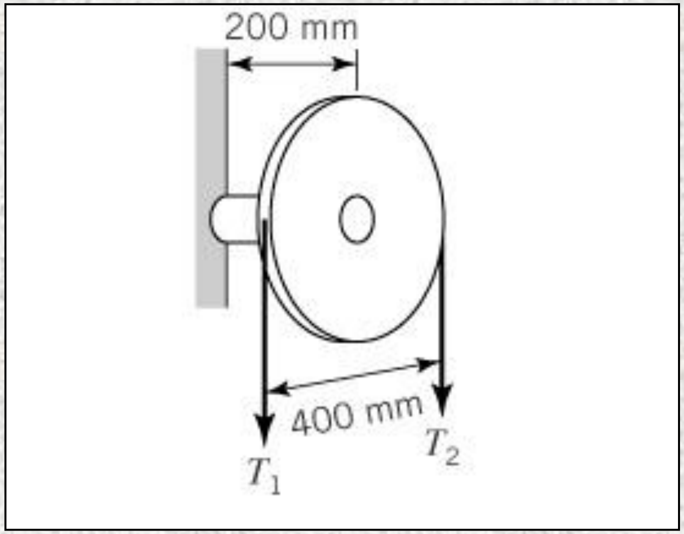


Figure c

Problem 4.40

4.40 A solid aluminum alloy (Y= 320 Mpa) shaft extends 200mm from a bearing support to the center of a 400 mm diameter pulley. The belt tensions T_1 and T_2 vary in magnitude with time. Their maximum values of the belt tensions are applied only a few times during the life of the shaft, determine the required diameter of the shaft if the factor of safety is SF= 2.20



Solution:

4.40 $M = 200(1800+180) = 396{,}000$ N.mm; $T = 200(1800-180) = 324{,}000$ N.mm

$$\sigma = SF\frac{Mc}{I} = \frac{2.20(396{,}000)(d)(64)}{2\pi d^4} = \frac{8{,}874{,}000}{d^3} \text{ (MPa)}$$

$$\tau = SF\frac{Tc}{J} = \frac{2.20(324{,}000)(d)(32)}{2\pi d^4} = \frac{3{,}630{,}000}{d^3} \text{ (MPa)}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{Y}{2} = \frac{320}{2} = \frac{1}{d^3}\sqrt{\left(\frac{8{,}874{,}000}{2}\right)^2 + (3{,}630{,}000)^2}$$

$d = \underline{32.97 \text{ mm}}$